6.5/6.6 : Delta Function & Convolution

CONSIDER $my'' + \gamma y' + ky = f(t)$, $\overbrace{y(0)=0,\ y'(0)=0}^{\text{START AT REST}}$

$$m\left(s^2\mathcal{L}\{y\} - sy(0) - y'(0)\right) + \gamma\left(s\mathcal{L}\{y\} - y(0)\right) + k\mathcal{L}\{y\} = \mathcal{L}\{f(t)\}$$

$$\Rightarrow (ms^2 + \gamma s + k)\mathcal{L}\{y\} = \mathcal{L}\{f(t)\}$$

$$\Rightarrow \mathcal{L}\{y\} = \frac{1}{ms^2+\gamma s+k}\mathcal{L}\{f(t)\}$$

$$\underbrace{Y(s)}_{\text{OUTPUT}} = \underbrace{G(s)}_{\text{TRANSFER}}\ \underbrace{F(s)}_{\text{INPUT}}$$

$\boxed{\text{Terminology from engineering}}$

$F(s) = \mathcal{L}\{f(t)\} =$ "the <u>input</u>" , $f(t) =$ 'forcing function'

$G(s) = \frac{1}{ms^2+\gamma s+k} =$ "the <u>transfer function</u>", $g(t) = \mathcal{L}^{-1}\{G(s)\}$ = "<u>impulse response</u>"

$Y(s) = \mathcal{L}\{y\} =$ "the <u>output</u>", $y(t) = \mathcal{L}^{-1}\{Y(s)\}$ = the <u>solution</u>

<u>Ex</u> $y'' + 4y = \cos(7t)$, $y(0)=0,\ y'(0)=0$

$$\Rightarrow (s^2+4)\mathcal{L}\{y\} = \mathcal{L}\{\cos(7t)\} \Rightarrow \mathcal{L}\{y\} = \frac{1}{s^2+4}\mathcal{L}\{\cos(7t)\}$$

$F(s) = \mathcal{L}\{\cos(7t)\} = \frac{s}{s^2+49}$ = "INPUT" $\rightarrow f(t) = \cos(7t)$

$G(s) = \frac{1}{s^2+4}$ = "TRANSFER FUNCTION" $\rightarrow g(t) = ?$

$Y(s) = G(s)F(s) = \frac{1}{s^2+4}\,\frac{s}{s^2+49}$ = "OUTPUT" $\rightarrow y(t) = ?$

$g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\sin(2t)$ = "impulse response"

$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s^2+4)}\,\frac{s}{(s^2+49)}\right\} = ??$ sol'n

WOULD HAVE TO DO PARTIAL FRACTIONS!

Convolution Integral Theorem

If $\mathcal{L}\{g(t)\} = G(s)$ and $\mathcal{L}\{f(t)\} = F(s)$

then $\mathcal{L}^{-1}\{G(s)F(s)\} = \underbrace{\int_0^t g(t-s)f(s)\,ds}_{\text{convolution integral.}}$

proof in handout

Ex) $y'' + 4y = \cos(7t)$, $y(0) = 0$, $y'(0) = 0$

- $Y(s) = G(s)F(s) = \dfrac{1}{(s^2+4)} \cdot \dfrac{s}{(s^2+49)}$
- $g(t) = \frac{1}{2}\sin(2t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$
- $f(t) = \cos(7t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+49}\right\}$

WE STOP HERE

$$\boxed{y(t) = \mathcal{L}^{-1}\{G(s)F(s)\} = \int_0^t \frac{1}{2}\sin(2(t-s))\cos(7s)\,ds}$$

USING A SYMBOLIC INTEGRATER WE GET

$\frac{1}{45}\cos(2t) - \frac{1}{45}\cos(7t)$ ("BEATS")

One cool feature, easy to quickly get the sol'n if the forcing function changes!

Ex) $y'' + 4y = f(t)$, $y(0) = 0$, $y'(0) = 0$

you give me ANY function!!!

$\Rightarrow y(t) = \int_0^t \frac{1}{2}\sin(2(t-s))\,f(s)\,ds$

7-11 like this

$\boxed{\text{HW } 10/7}$ $y'' + 6y' + 9y = f(t)$, $y(0) = 0$, $y'(0) = 0$

$G(s) = \dfrac{1}{s^2+6s+9} = \dfrac{1}{(s+3)^2}$ ← TRANSFER FUNCTION

$\mathcal{L}^{-1}\{G(s)\} = t e^{-3t} = g(t)$

$(t-\tau)$ use τ instead of s

$$y(t) = \int_0^t \underbrace{\left((t-\tau)e^{-3(t-\tau)}\right)}_{\text{Impulse response!}} f(\tau)\,d\tau$$

The Delta Function

Why do we call $g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{ms^2+\gamma s+k}\right\}$ the "impulse response"?

$$\boxed{my'' + \gamma y' + ky = f(t), \quad y(0) = 0, \ y'(0) = 0}$$

$$Y(s) = G(s)F(s)$$

IF $F(s) = 1$, THEN $Y(s) = G(s)$,

which means

$y(t) =$ "the sol'n" $= \mathcal{L}^{-1}\{G(s)\} = g(t) =$ "impulse response"

THUS, $g(t)$ CAN BE THOUGHT OF AS THE SOLUTION TO ABOVE SYSTEM WHEN $\mathcal{L}\{f(t)\} = 1$.

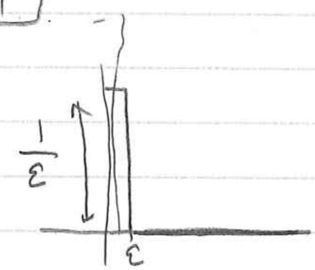
BUT THIS IS NOT IN OUR TABLE!

WE CALL THIS $\delta_0(t) \longrightarrow$???

"DIRAC DELTA FUNCTION"

DERIVATION, APPROACHING $\mathcal{L}\{f(t)\} = 1$

Let $h_\varepsilon(t) = \begin{cases} 1/\varepsilon, & t \le \varepsilon; \\ 0, & t > \varepsilon. \end{cases}$



$$\mathcal{L}\{h_\varepsilon(t)\} = \int_0^\infty e^{-st} h_\varepsilon(t)\,dt$$

$$= \int_0^\varepsilon e^{-st}\frac{1}{\varepsilon}\,dt = \frac{1}{\varepsilon}\frac{(1-e^{-\varepsilon s})}{s} = \frac{(1-e^{-\varepsilon s})}{\varepsilon s}$$

$$\lim_{\varepsilon\to 0}\left(\mathcal{L}\{h_\varepsilon(t)\}\right) = \lim_{\varepsilon\to 0}\frac{(1-e^{-\varepsilon s})}{\varepsilon s} \overset{\frac{0}{0}}{=} \lim_{\varepsilon\to 0}\frac{se^{-\varepsilon s}}{s} = 1$$

TYPO IN REVIEW

As $\varepsilon \to 0$, $h_\varepsilon(t) =$ "VERY LARGE VALUE FOR EXTREMELY SHORT TIME, THEN ZERO"

We define $\delta_0(t)$ SUCH THAT

$\mathcal{L}\{\delta_0(t)\} = 1$ AND WE THINK OF

$\delta_0(t)$ AS THE "FUNCTION" $h_\varepsilon(t)$ APPROACHES AS $\varepsilon \to 0$.

Similarly, $\delta_c(t)$ is an "impulse" at $t=c$.

$h_\varepsilon(t) = \frac{1}{\varepsilon}\left(u_{c-\varepsilon}(t) - u_{c+\varepsilon}(t)\right)$

and

$$\mathcal{L}\{\delta_c(t)\} = \lim_{\varepsilon \to 0} \int_0^\infty e^{-st} h_\varepsilon(t)\,dt$$

$$= e^{-cs}$$

SEE NOTES

Note

$$\mathcal{L}\{\delta_0(t)\} = e^{-0s} = 1$$

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with graph → Ex] $y'' + 4y = \delta_3(t)$ $\quad y(0)=0,\ y'(0)=0$

$$\Rightarrow (s^2+4)\mathcal{L}\{y\} = \mathcal{L}\{\delta_3(t)\}$$

$$\mathcal{L}\{y\} = \frac{1}{s^2+4} e^{-3s}$$

$$y = \mathcal{L}^{-1}\left\{e^{-3s}\frac{1}{s^2+4}\right\}$$

$$= u_3(t)\, \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}\Big|_{t\to t-3}$$

impulse response

$$= u_3(t)\left(\frac{1}{2}\sin(2t)\Big|_{t\to t-3}\right)$$

$$y = u_3(t)\frac{1}{2}\sin(2(t-3)) = \begin{cases} 0, & t<3; \\ \frac{1}{2}\sin(2(t-3)), & t \geq 3. \end{cases}$$

"impulse"

Ex) $y'' + 4y' + 40y = \delta_0(t)$, $y(0) = 0, y'(0) = 0$

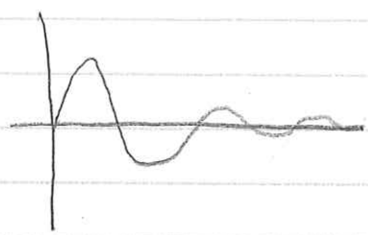
$$(s^2 + 4s + 40)\mathcal{L}\{y\} = \mathcal{L}\{\delta_0(t)\} = 1$$

$$\mathcal{L}\{y\} = \frac{1}{s^2+4s+40} = \frac{1}{s^2+4s+4+36}$$

$$= \frac{1}{(s+2)^2+36}$$

$$= \frac{1}{6}\,\frac{6}{(s+2)^2+36}$$

$$\Rightarrow y = \frac{1}{6}e^{-2t}\sin(6t) = \text{"impulse response"}$$



Ex) $y'' + 4y' + 40y = \delta_7(t)$

$$\Rightarrow \quad \mathcal{L}\{y\} = e^{-7s}\,\frac{1}{6}\,\frac{6}{(s+2)^2+36}$$

$$\Rightarrow \quad y = u_7(t)\left(\frac{1}{6}e^{-2t}\sin(6t)\right)\Big|_{t\to t-7}$$

$$= \boxed{u_7(t)\,\frac{1}{6}e^{-2(t-7)}\sin(6(t-7))}$$